FORMAL PROPERTIES OF THE MATCHING LAW[1]

R. J. HERRNSTEIN

HARVARD UNIVERSITY

The matching law implies that any form of behavior approaches an asymptotic frequency as its reinforcement approaches 100 per cent of the total reinforcement being obtained at a given time. This asymptote is formally independent of the kind or quantity of drive or reinforcement associated with the response in question or with any competing response.

It has been repeatedly shown that animals (including human beings) distribute their behavior across alternatives according to the matching principle (Herrnstein, 1970, 1971). Given equivalent response forms and reinforcements varying in quantity, an animal's responses obey the rule:

$$\frac{P_1}{P_1 + P_2 + \ldots + P_N} = \frac{R_1}{R_1 + R_2 + \ldots + R_N} \tag{1}$$

P tallies responses; R tallies reinforcements. The subscripts identify responses with the reinforcements they produce. Rachlin (1971) and Herrnstein (1971) noted that even when responses or reinforcements differ qualitatively, matching may still describe choice if the measures of each are suitably weighted. Thus, at least in principle, matching appears to be a general formal framework to replace the traditionally informal (or non-formal) vocabulary for the law of effect.

Equation 1 may not be the most fundamental way to describe behavior, simply because it focuses on a derived measure, relative responding. A matching of ratios of responses and reinforcements, while significant in its own right, can be invariant over changes in the absolute level of responding or reinforce-

ment. Thus, as stated above, the matching principle fails to make explicit what might happen with changes in drive or in response effort or, generally, with changes in the influence of variables external to the particular responses and reinforcements under observation. The mass of evidence suggests that the absolute rate of each response reflects the status of all reinforcements acting at a given time on a given subject as follows:

$$P_1 = \frac{kR_1}{\sum_{i=0}^{n} R_i} \tag{2}$$

Here, P_1 refers to the number of responses over some unit of time, R_1 refers to the number of reinforcements delivered for occurrences of that response, and the denominator expresses the total amount of the reinforcement.

The central purpose of this paper is to show what is logically entailed by equations 1 and 2, i.e., by the matching law as it is ordinarily expressed. Recent variations in the formal descriptions of matching (e.g., Baum, 1973) do not alter the conclusions to be drawn here, although the variations may deserve note on their own terms. In this paper, no attempt is made to evaluate empirically the matching law itself nor any of its closely related variants, the assumption being that equations 1 and 2 have had a degree of verification that justifies an effort to explore their logical properties at this point. One reason for this sort of exercise is to expose the matching law's vulnerabilities to the future tests of data, but here only the tacit logical frame-

work is set forth. The logical structure hinges on the parameters k and n in equation 2, the interpretation of which is briefly set aside, while subsidiary matters (given present purposes) are disposed of.

Equation 2 presupposes that the subject we are describing correlates P_1 with its reinforcement, R_1, the same way we do—i.e., the pigeon "knows" its pecks bring the food; the rat "knows" the lever press avoids the shock, and so on. This usually poses no problem, for in our experiments we typically observe animals doing things they would not do except for the reinforcements we provide, and we usually make sure that the correlation between response and reinforcement is clear. It is worth noting, however, that animals (and people) in nature may often be quite confused over what causes what, and they may also place value (i.e., reinforcing value) on responses in and of themselves, independent of extrinsic consequences. While those facts surely make extrapolation hazardous, they do not necessarily limit the analysis within its present boundaries.

In contrast to R_1, we rarely if ever know the true value of the denominator, ΣR_i, even in experiments in which extraneous sources of reinforcement have been screened out. One source of uncertainty arises from the choice of an interval of time over which to summate reinforcements. We know from the contrast effects in multiple schedules that reinforcements spread their effect, albeit with decrement, into periods of time during which they did not and cannot occur (Herrnstein, 1970). The transition from high to low rates of reinforcement, and vice versa, tend to be psychologically gradual, no matter how abrupt they are objectively. As a result of this, the denominator may often be changing continuously even under objectively fixed circumstances.

A more serious, because more inescapable, problem with the denominator is that animals find unanticipated reinforcement in virtually any environment, no matter how hard the experimenter has tried to make it barren. A creature's own body and its physiological processes introduce a background level of hedonic affect that puts the denominator beyond total control.

Note, finally, that the summation extends from R_o upwards[2]. This notation is used to

suggest that there may be reinforcements that are not conditional upon responses, and which therefore come independently of action (and for which we could not therefore find an instrumental response even if we looked). The denominator includes all reinforcements, even the spontaneous ones. The correct interpretation of n, therefore, is that it enumerates the instrumental sources of reinforcement—those dependent upon responses to produce them. The total number of sources, counting spontaneous reinforcements as a single source, is $n + 1$.

The uncertainties in the denominator ordinarily cause no trouble in the prediction of choices among specified responses, P_1, P_2, or any other number. An earlier paper (Herrnstein, 1970) showed how the unknown denominator disappears from view in the usual choice procedure. A brief recapitulation follows:

(a)
$$P_1 = \frac{kR_1}{\sum_{i=0}^{n} R_i}$$

(b)
$$P_2 = \frac{kR_2}{\sum_{i=0}^{n} R_i} \qquad (3)$$

(c)
$$\frac{P_1}{P_1 + P_2} = \frac{\dfrac{kR_1}{\Sigma R_i}}{\dfrac{kR_1}{\Sigma R_i} + \dfrac{kR_2}{\Sigma R_i}} = \frac{R_1}{R_1 + R_2}$$

This sequence of equations derives matching by applying the general form of equation 2 to each of a pair of choices. Note that matching thus entails the invariance of k for each response, whatever its reinforcement. Were this not so, we would not be able to cancel out the k's in equation 3(c). Matching also entails the invariance of k with respect to the reinforcement for alternative responses: for example k in equation 3(a) is not taken to be dependent upon R_2, nor is k in equation 3(b) taken to be dependent on R_1. This line of argument could obviously be generalized to

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[2] R_o here has the narrow meaning of reinforcements that come spontaneously. In other papers, I, and other workers, have used the expression to mean all reinforcements except R_1, that is, $\Sigma R_i - R_1 = R_o$. It seems more convenient to adopt the restrictive meaning, even at the risk of some confusion.

every R_i in ΣR_i, which is tantamount to saying that if equation 2 is to subsume matching, it must assume invariance of k with changes in any reinforcement. This invariance of k is the crucial formal characteristic of the matching law.

To return to equation 3(c), so long as each response being observed has the same k and the same denominator, the choice between them conforms to simple matching—equation 1 above—no matter how complex and varied the unmeasured sources of reinforcement besides R_1 and R_2. The absolute rates of P_1 and P_2 would, of course, respond to changes in the denominator, but absolute rates usually get disregarded in experiments on choice. Indeed, standard choice procedures outside the operant tradition use discrete trials, which trivializes variations in absolute rates of responding.

We are now ready to address ourselves directly to the main issue, which is the formal status of k in equation 2. What sort of thing is k, aside from its role as a curve-fitting parameter in experiments relating response rate to reinforcement rate? It is clear that k sets an upper bound to the rate of P_1, obtained[3] when $R_1 = \Sigma R_i$, which is to say under the unlikely circumstance that all the reinforcement being obtained by the subject is the reinforcement correlated with P_1. Therefore, k, having the same dimensions as P_1, is just a response rate. But asymptotic rate of response is not the only thing we know about k, because of the matching principle itself.

By transposing from equation 2, we find that:

$$\frac{P_1}{k} = \frac{R_1}{\sum_{i=0}^{n} R_i} \qquad (4)$$

Note that equation 4 has the same form as equation 1, a relative frequency of response equalling a relative frequency of reinforcement. Insofar as the matching principle holds in equation 4 as in equation 1—and there is no reason to suppose otherwise—we may infer that k is a frequency of behavior such that

when divided into the frequency of the observed response, P_1, matching is found. As an empirical matter, this statement carries little weight, for although P_1 divided by k should equal R_1 divided by ΣR_i, we do not usually know ΣR_i. However, it is offered here not so much as an empirical prediction, but as a logical property of the parameter k given the underlying relations.

In addition, by combining equations 1 and 4, it follows that:

$$k = \sum_{i=0}^{n} P_i \qquad (5)$$

Over any specified collection of choices, the denominators on either side of the matching equation have corresponding terms for each subscript (see equation 1), so that k must include a term for every source of reinforcement. k is, then, first of all the asymptote of the observed response; secondly, it is a quantity such that the observed response divided into it would obey matching if we knew total reinforcements, and, finally, it is the total amount of behavior generated by all the reinforcements operating on the subject at a given time. But, regarding the last point, note that k includes P_o, which corresponds to the spontaneous reinforcements, R_o. When there are such reinforcements, we must reckon with some quantity of behavior used up, as it were, in experiencing them, which would keep any observed response from attaining its asymptote.

It may strike some as frivolous to assume that we must talk about responses consumed by the obtaining of spontaneous reinforcements, which by definition are independent of any correlation with response. Nevertheless, the logic of matching once again leaves no alternative. It is clear that the subject's response cannot reach asymptote so long as $R_1 \neq \Sigma R_i$ (see equation 2) even when the extra reinforcements are spontaneous. We can see the necessity for P_o more clearly if we imagine a simple situation in which there are, let us suppose, just two response alternatives P_1 and P_2, producing reinforcements R_1 and R_2.

When there are no spontaneous reinforcements (R_o), responding is accounted for by two equations and their corollary given matching:

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[3]This overlooks the possibility that $R_1 > \Sigma R_i$, which would be the case if the aggregate reinforcement exclusive of R_1 were negative. It is not yet clear how the contribution of punishers is to be added to the present formulation.

(a) $\qquad P_1 = \dfrac{kR_1}{R_1 + R_2}$

(b) $\qquad P_2 = \dfrac{kR_2}{R_1 + R_2}$

∴ (c) $\qquad k = P_1 + P_2 \qquad$ (6)

This assumes that P_1 and P_2 have the same k (an assumption that does not affect the generality of the argument, as we will see later) and it conforms to the matching relation, as assumed. Now let us suppose that we add a spontaneous source of reinforcement R_0, while holding R_1 and R_2 constant. From equations 6(a) and 6(b), we see that P_1 and P_2 must decline. But k can no longer be just the sum of those two responses as in equation 6(c), for if it were, the matching relation would be violated.

If k were still the sum of P_1 and P_2, then $(P_1 + P_2)/k = 1$. But, given R_0 and expressing relative responding and reinforcement in the form prescribed by equation 4:

$$\frac{P_1 + P_2}{k} = \frac{R_1 + R_2}{R_1 + R_2 + R_0} \neq 1.0; \qquad (7)$$

which is contrary to the supposition above. k must therefore contain some additional quantity, P_0, such that the matching in equation 7 is fulfilled. The asymptote for any response does not vary with the reinforcement for competing responses, as noted above. In the absence of any special assumptions, the invariance of k would also apply to the effects of R_0. Thus, k would remain constant even as P_1 and P_2 fell, the slack being taken up by P_0.

If k, then, is the total amount of behavior, including any P_0, what interpretation should be placed on its value for a subject? Are we, for example, to suppose that the 60 to 110 responses per minute found for the pecking response (Herrnstein, 1970) tells us that the pigeons are doing just so many things a minute? The answer is yes, but there is a crucial qualification. It is 60 to 110 things a minute *scaled with respect to the observed response,* pecking. Throughout this discussion it has been tacitly assumed that the rule for tallying all sorts of responses makes them directly commensurable. In effect, commensurability means that the rule for tallying responses must make their frequencies equal when the

reinforcements for them are equal and also to match at all other frequencies of reinforcement. Thus, if the frequency of reinforcement for pecking happens to be half the quantity in the denominator of equation 2, then k should equal $2P_1$. But $k = 2P_1$ only if that part of k that is not pecking is counted in peck-units. In practice, it may be quite hard to transform the raw counts of behavior into suitable units, but that does not alter the theoretical requirement that the value of k should be equal to the number of responses we would obtain if all reinforcements went to pecking alone.

One might suppose that the size of k must reflect the total amount of reinforcement operating on an animal. After all, it might seem that if the total amount of reinforcement goes up, then the total amount of behavior, however measured, should go up too. But the conclusion is faulty, given the equations. In this scheme, only relative reinforcement counts. Nowhere is there anything relating behavior to an absolute frequency of reinforcement. And since the total amount of reinforcement (ΣR_i) is just a sum of absolute frequencies, it can exert no influence of its own. Equation 2 shows that the asymptote reached by any form of response is independent of the total amount of reinforcement. It makes no difference what the size of ΣR_i is when R_1 overtakes it, for at that point the reinforcements in the equation cancel out. Similarly, even before any response reaches its asymptotic level, any change in the total reinforcement must have a net effect of zero on the total behavior, which is to say, it leaves k invariant, as already shown.

Consider again equation 4. The subject is engaging in P_1 at some rate and receiving R_1 reinforcements. With the background reinforcement, there is some level of total reinforcement, ΣR_i, which actually determines what fraction of k emerges in the form of P_1. Now we increase ΣR_i by increasing the amount of reinforcement from sources other than R_1. The observed result is a decrement in P_1 (Rachlin and Baum, 1972), in magnitude appropriate to equation 2. The change in P_1 must, however, be exactly compensated for by changes in other responses, for otherwise k would change with any and every change in total reinforcement. Matching as we know it could not occur if k changed when-

ever ΣR_i changed, for then the series of equations under (3) would break down. If the changes in the various responses did not balance out entirely, then it could no longer be said that the value of k is the asymptote of P_1 as it absorbs all reinforcement, independent of the total amount of reinforcement. And if that were no longer true, then none of the foregoing would be correct either. The balancing off of the effects of changes in total reinforcement, leaving k invariant, is, then, a necessary, logical consequence of the matching principle as it has been understood. Of course, the compensating changes in responses need to be tallied in units commensurate with P_1, for the reasons already given.

Summarizing the main points to here, it has been shown that k is dimensionally just a measure of behavior, such as "responses per minute". Moreover, it is the amount of behavior that the observed response would display if there were no source of reinforcement other than the one associated with the observed response. When there are other sources of reinforcement, then k measures the total frequency of all responses in units commensurate with the measure of the observed response. Changes in the over-all frequency of reinforcement leave k invariant, since the sole determining influence on the size of k is the response form itself, without regard to the amount or type of reinforcement conditional upon it or on anything else.

The proper interpretation of k is therefore as the modulus for measuring behavior. In a situation in which an animal can make various responses and receive reinforcement, the value of k depends entirely on which response is taken as P_1. If we take a response that has a high asymptotic frequency, then the total amount of behavior measured in its units will be large. However, if we take a response whose asymptotic frequency is low, then the total amount of behavior will be small. The amount of behavior in the two cases may be the same with respect to an external criterion, but the modulus is changed. Why some responses have high or low asymptotes may be an interesting empirical question in its own right, but beside the point. The present interest is focussed on the interpretation of the key parameter in the matching law, which turns out to be remarkable only in its simplicity.

With the present interpretation of k firmly in mind, let us consider the general form of the matching law, equation 2, a bit further. The denominator, ΣR_i, sums across all reinforcements. The reinforcements may come from different drives, in varying sizes and potencies. Just as we needed to express responses in commensurate form, we also need to express reinforcers commensurately. This means that we must adjust our tallies of reinforcers such that matching will be preserved. A drop of water may be worth as much as, more than, or less than, a pellet of food. The sole criterion is the distribution of behavior across choices between them. Knowing that, we can count reinforcers in units equivalent to R_1, whichever we take as such. It obviously makes no difference whether we say 1 pellet = 0.5 drops or 1 drop = 2 pellets. Now suppose we change the drive applicable to R_1, let us say by making the subject less hungry. As far as equation 2 is concerned, the change is indistinguishable from a reduction in the magnitude of R_1. The predicted outcome is a reduction in P_1. But the reduction depends not just on the effective change in R_1; it also depends on the size of k and ΣR_i, for behavior always depends upon the entire context of reinforcement and on the asymptote, k. To be sure, we usually do not know how much a given change in body weight, for example, changes the reinforcing power of a pellet. A reduction in hunger may be formally equivalent to a decrease in pellet size but, empirically, pellet sizes are easy to gauge while changes in drive are forbiddingly obscure, at least they are obscure before the fact.

They are obscure because a change in reinforcing power reveals itself only relative to other reinforcers. A reduction in the reinforcing power of a pellet means that our modulus for measuring reinforcement has shrunk. Ten reinforcements per hour of food now equals some smaller number of other reinforcements than it did previously. If the other reinforcements coming to the animal have not changed, their total impact will add up to a *larger* number expressed in the shrunken modulus. For example, if a pellet of food had previously equalled a drop of water, it may now equal 0.5 drops of water. If thirst and the rate of water reinforcement stay put, the reinforcement derived from water is now ex-

pressed by a number twice as large. This assumes that changing hunger does not change thirst *per se,* and that only the modulus is changed. In actual fact, drives do interact—especially hunger and thirst—but once again, it is only the structural aspects of the argument that engage our attention here.

Whether ΣR_i gets larger or smaller, or even remains unchanged, with any given change in drive applicable to R_1 depends upon the other terms in ΣR_i. For example, if hunger were the only drive acting on an animal (which means that all the reinforcers would expand or contract together) then a change in drive would alter nothing. Not even the absolute levels of behavior would change, for each response would continue to command a given fraction of the total reinforcement, and it is only the fraction of the total reinforcement that controls responding (see equation 2). Furthermore, as already shown, nothing we do to reinforcement alters k, so that changes in drive, however diverse and intense, leave k invariant. All that can be said generally about changes in drive is that they are detectable only against a background of behavior sustained by other drives.

The present interpretation may seem to sidestep significant issues in the analysis of behavior under the control of the law of effect. For example, one might want to know whether a creature behaves more in one situation than another. But that question presupposes a natural, absolute scale for behavior that makes separate situations comparable, a supposition without basis. Actually, there are only various moduli, some large and some

small numerically, and the total behavior (including P_o) in *any* situation is entirely predetermined by the choice of modulus, however rich or poor the reinforcement or keen or slack the drives. If this seems wrong, then the fault must be in the matching law.

We can, of course, compare the k's obtained with different responses, within or across situations (it makes no difference). The list of k's for an animal's various topographies tells us something about the topographies themselves. The comparison may show that the only factor influencing k is the physical constraint of time (Baum, 1973), since some responses take longer to execute than others. Or, the constraints may not be so obviously physical. In any event, the formal constancy of k and its corollaries are an implication of the matching law that seems especially vulnerable to empirical confirmation or refutation.

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